Sensitivity Analysis

The purpose of this document is two-fold, to show how we go beyond significant digits to show uncertainty more precisely, and to show how to use uncertainties in calculations.

A Notation for Uncertainty

Our text defines **absolute error** as “measured value – true value”. Mathematical studies, however, cannot be extended very far without some sort of symbolic notation. Let’s use the term “actual value” rather than “true value” and introduce symbols X_M and X_A, respectively, to denote measured value and actual value. The difference is denoted by $\boldsymbol{\Delta x}$. The symbol Δ is the upper-case Greek letter *delta*, which is often used in mathematics to denote a change in something or a difference between two values. The symbol $\boldsymbol{\Delta x}$, then, is read “delta x”, and we express the absolute measurement error as

$$\Delta x = X_M - X_A$$

Don’t be too concerned if you’ve seen the difference reversed, *i.e.* $\Delta x = X_A - X_M$. It’s not a very important distinction, since most errors can be either positive or negative. We’ll simply follow the textbook’s lead. Using symbols, we see that **relative error** can be expressed by the ratio

$$\frac{\Delta x}{X_A}$$

Relative error is also known as **percent error** if the ratio is expressed as a percent.

Example 1

Suppose we want to estimate the dimensions of a room by counting the number of 1-foot square floor tiles along the front and side walls. Imagine that we count 25 tiles plus a small part of another tile along the front wall and 28 tiles plus almost another whole tile along the side wall. So we report that the room is 25 feet wide by 29 feet long. That is, for the width, $X_M = 25$ feet, and, for the length, $X_M = 29$ feet.

For each dimension, there is, of course, some actual value, perhaps $25\,{}^1\!/_4$ feet and $28\,{}^7\!/_8$ feet, let us say. Then, the absolute errors are $\Delta x = 25\ \mathit{feet} - 25\,{}^1\!/_4\ \mathit{feet} = -{}^1\!/_4\ \mathit{foot}$ for the width and $\Delta x = 29\ \mathit{feet} - 28\,{}^7\!/_8\ \mathit{feet} = {}^1\!/_8\ \mathit{foot}$ for the length. The percent errors are, first for the width, $\dfrac{\Delta x}{X_A} = \dfrac{-{}^1\!/_4\ \mathit{foot}}{25\,{}^1\!/_4\ \mathit{feet}} = -0.\overline{0099} \approx -1.0\%$, and then for the length, $\dfrac{\Delta x}{X_A} = \dfrac{{}^1\!/_8\ \mathit{foot}}{28\,{}^7\!/_8\ \mathit{feet}} = 0.\overline{004329} \approx 0.4\%$.

In this example we were able to calculate $\boldsymbol{\Delta x}$ based on X_M only because we were told X_A. How likely is that? Why would we measure X_M if we know X_A? What occurs more often is a situation where we measure a value X_M and have a reasonable idea of the accuracy limits of our measuring device. In Example 1, we expect our measured distances to be accurate to within 1 foot since that is the size of the floor tiles. It’s as if we used a measuring device with “tick marks” 1 foot apart. Let us define, then, a more useful quantity.

> The **Measurement Uncertainty** or the **Magnitude of the Error** is the absolute value of the maximum measurement error that can occur based on the accuracy of the measuring device. We denote the magnitude of the error by $\boldsymbol{\Delta X}$ (capital “*X*”).

Because this definition is in terms of an absolute value, we see that $\boldsymbol{\Delta X}$ is always positive or zero and that $|\Delta x| \le \Delta X$ for any measurement. That means that both $\boldsymbol{\Delta x}$ and $-\boldsymbol{\Delta x}$ lie in the interval between $-\boldsymbol{\Delta X}$ and $\boldsymbol{\Delta X}$. Using mathematical symbols,

$$\begin{array}{cc} -\Delta X \le \Delta x \le \Delta X & -\Delta X \le -\Delta x \le \Delta X \\ -\Delta X \le X_M - X_A \le \Delta X & -\Delta X \le X_A - X_M \le \Delta X \\ X_A - \Delta X \le X_M \le X_A + \Delta X & X_M - \Delta X \le X_A \le X_M + \Delta X \end{array}$$

Thus, any measured value is bracketed by an interval centered on the actual value, from $X_A - \Delta X$ to $X_A + \Delta X$. Furthermore, for any measured value, the actual value is bracketed by an interval centered on the measured value, from $X_M - \Delta X$ to $X_M + \Delta X$.

Example 2

Suppose someone uses a yardstick with tick marks 1/8 inch apart and reports the length of a board as 32½ inches. Find a bracketing interval for the actual length of the board.

Solution:

Here X_M = 32½ inches and ΔX = 1/8 inch. So we know X_A lies between 32½ inches - 1/8 inch = 32 3/8 inches and 32½ inches + 1/8 inch = 32 5/8 inches. That is the bracketing interval, expressed symbolically as

$$32\ 3/8 \text{ inches} \le X_A \le 32\ 5/8 \text{ inches}$$

Measurement uncertainty is not always stated as the separation of tick marks on a device. Sometimes we are just told the value. For example, "The machine weighed 150 kilograms plus or minus 3 kilograms." This statement tells us X_M = 150 kg and ΔX = 3 kg. Sometimes this is expressed as (150 ± 3) kg or 150 kg ± 3 kg. Regardless of how the information is presented, we know that the actual weight is between 147 kg and 153 kg: 147 kg $\le X_A \le$ 153 kg.

Just as **relative error** or **percent error** is usually more meaningful than **absolute error**, so too is **percentage uncertainty**, the ratio of **measurement uncertainty** to **measured value**, more meaningful than the measurement uncertainty itself.

$$\textbf{Percentage Uncertainty} = \frac{\Delta X}{X_M} = \frac{\text{Magnitude of the error}}{\text{Measured value}}$$

Consider the next example, where this value is reported along with the measured value.

Example 3

"The fish weighed 14.3 pounds with a 2% uncertainty." (Or, "The fish weighed 14.3 pounds ± 2%"). Find a bracketing interval for the actual weight of the fish.

Solution:

In this case the percentage uncertainty is give as 2% and X_M = 14.3 pounds. So

$$2\% = 0.02 = \frac{\Delta X}{X_M} = \frac{\Delta X}{14.3\,lbs}$$

$$\Delta X = (0.02) \times (14.3\,lbs) = 0.286\,lbs$$

Thus a bracketing interval for the actual weight is (14.3 ± 0.286) *lbs* or $14.014\,lbs \le X_A \le 14.586\,lbs$.

Exercises

1. The age of a wood fragment from an Egyptian pyramid has been measured by radiocarbon dating. Based on the report, the bracket of ages in which we would expect to find the true age of the fragment is 3350 years $\le X_A \le$ 3750 years. (a) What age was reported? (b) What measurement uncertainty? (c) What percent uncertainty?

2. In measuring the speed of sound in dry air the testing team posted the result as 749.1 mph. However, it admitted that its measuring device had a measurement uncertainty of 15 mph. (a) Bracket the actual speed of sound in dry air based on this report. (b) What is the percentage uncertainty in their reported result?

3. A thermometer to measure body temperature has tick marks every one-fifth of a degree Fahrenheit. A nurse using this thermometer reports the temperature of a patient to be 102.4°F. (a) What is the measurement uncertainty in this value? (b) What interval brackets the patient's actual temperature? (c) What is the percent uncertainty in the reported temperature?

4. (a) Find a bracket for the exact weight of an object whose reported weight is 5.21 kg ± 3%. (b) What is the magnitude of the error in this report?

Sensitivity Analysis

Sensitivity analysis is a general term that refers to analyzing the effect of measurement errors on results computed from the measurements.

Example 4

(a) What is the area of the room that we measured in Example 1? (b) What is the bracketing interval for the area? (c) What is the magnitude of the error in area? (d) What is the percentage uncertainty?

Solution:

(a) Recall the dimensions we reported were 25 feet wide by 29 feet long, with a magnitude of error of 1 foot in each dimension. If we use 25 feet and 29 feet to compute the area of the room, we get 725 square feet. That's certainly the simplest value to obtain, and many people would choose to report that result. However, as we'll see, this might not be the best choice. (b) Now let's find the bracketing interval for the area. Since we could be off by at most 1 foot in each measurement, the front wall could be anywhere from 24 to 26 feet long, and the side wall could be anywhere from 28 to 30 feet long. If we use the smaller value for each dimension, we get $24\,\mathit{feet} \times 28\,\mathit{feet} = 672\,\mathit{square\ feet}$. On the other hand, if we use the larger values we get $26\,\mathit{feet} \times 30\,\mathit{feet} = 780\,\mathit{square\ feet}$. Thus we conclude that a bracket for the actual area of the room would be 672 square feet to 780 square feet. This answers (b), but let's take another look at our answer to (a). We should check whether the area of 725 square feet, computed above, lies at the midpoint of this interval, since any reported value so far has been at the midpoint of its bracketing interval. To find whether 725 square feet is the mid-point, take the average of the two extremes:

$\dfrac{672\,ft^2 + 780\,ft^2}{2} = \dfrac{1452\,ft}{2} = 726\,ft^2$!!!!! The interval that brackets the true area is centered on 726 ft^2, not 725 ft^2. This is typical of what happens when approximate data is used in formulas. Until the bracketing interval is computed, its midpoint is not known. And you will be expected, in this course, to use the midpoint of the interval as the reported value. That is the value that will be used in the next two parts of the question. (c) The magnitude of the error is the half-length of the bracketing interval: $\boldsymbol{\Delta X} = \frac{1}{2}(780\,ft^2 - 672\,ft^2) = 54\,ft^2$. (d) And, finally, the percentage uncertainty is obtained by dividing $\dfrac{54\,ft^2}{726\,ft^2} \approx 7.4\%$.

A question arises on how this percentage uncertainty compares to the percentage uncertainty in the individual values used in the calculation. Well, the percentage uncertainty in the 25-foot reported width is $\dfrac{1\,ft}{25\,ft} = 4.0\%$, and the percentage uncertainty in the 29-foot reported length is $\dfrac{1\,ft}{29\,ft} \approx 3.4\%$. Note what appears to be a coincidence—that 4% plus 3.4% is 7.4%. Actually, it can be shown in the calculus that *small* percentage uncertainties should be *added* when the variables they are associated with are *multiplied <u>or</u> divided*.

Example 5

With a half percent uncertainty, the radius of the Earth is 3958 miles. Discuss the volume of the Earth along the lines of the discussion in Example 4.

Solution:

The volume of the Earth would be calculated by using the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$. But we learned in Example 4 that we should not simply apply this formula using the stated radius. With a ½% uncertainty in the radius, we have $\boldsymbol{\Delta X} = (0.005) \times (3958\text{ miles}) = 19.79$ miles. Using the small end of the radius bracket (3838.21 miles) we get a volume of roughly 2.5585×10^{11} cubic miles. Using the top end of the radius bracket (3977.79 miles) gives a volume of roughly 2.6364×10^{11} cubic miles. The midpoint of that bracket is 2.59745×10^{11} cubic miles, and its half-length is $\boldsymbol{\Delta X} = 3.895 \times 10^9$ cubic miles. So, the volume of the Earth is 2.59745×10^{11} cubic miles, plus or minus 3.895×10^9 cubic miles. The percentage uncertainty is $\dfrac{3.895 \times 10^9\,mi^3}{2.59745 \times 10^{11}\,mi^3} \approx 1.500\%$. Here again we see that, corresponding to $\boldsymbol{r} \times \boldsymbol{r} \times \boldsymbol{r}$, the percentage uncertainty is ½% + ½% + ½% = 1½%.

Example 6

Jim rode his bicycle one mile (5280 feet), plus or minus 0.7%. It took him 6 minutes, 40 seconds, plus or minus 4 seconds. What was the percentage uncertainty in his speed?

Solution:

First let's use the approximation stated in Example 3, that *small* percentage uncertainties should be *added* when the variables they are associated with are *multiplied or divided*. Jim's time is 400 seconds, so his percentage uncertainty in time is 1%. Add that to the 0.7% uncertainty in distance and you have 1.7% uncertainty for the speed. Maybe you're not convinced. After all, we're dividing now and we were multiplying before. So we'll work through step by step. For distance, $0.7\% = 0.007 = \frac{\Delta X}{X_M} = \frac{\Delta X}{5280\,ft}$, so $\mathbf{\Delta X}$ = (0.007) × (5280 *ft*) ≈ 37.0 *ft*. For time, $\mathbf{\Delta X}$ = 4 seconds. What's different about having to *divide* distance by time is that the *smallest distance*, divided by the *largest time*, yields the *smallest speed*. And the *largest distance*, divided by the *smallest time*, yields the *largest speed*. So the bracketing interval for speed is $\frac{5243\,ft}{404\,\text{sec}} \approx 12.98\,\frac{ft}{s}$ to $\frac{5317\,ft}{396\,\text{sec}} \approx 13.43\,\frac{ft}{s}$. Clearly, for speed, the half-length of this interval is $\mathbf{\Delta X}$ = 0.225 feet per second, and the midpoint is approximately 13.20 feet per second. The percent uncertainty of speed is $\frac{0.225\,\frac{ft}{s}}{13.20\,\frac{ft}{s}} \approx 1.70\%$ just as predicted.

Exercises

1. Discuss (as in the foregoing examples) the volume of a large box whose measurements are 6 feet by 3.5 feet by 4 feet if each measurement has a magnitude of error of 0.5 feet.

2. Discuss (as above) the volume of the box in problem 1 if the magnitude of the error in each measurement was 0.1 foot.

3. Discuss (as above) the volume of the box in problem 1 if the percentage error in each measurement was 0.38%.

4. To calculate the speed of an object it was observed that it traveled 2800 feet ± 20 feet in 4.4 seconds ± 0.1 second. Recalling that rate (speed) equals distance divided by time, discuss (as above) the speed of the object.

5. Discuss (as above) the volume of a large right circular cone having a height of 14 feet ± 0.6 foot and a base radius of 7.6 feet ± 0.4 foot. The formula for the volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$, where *r* is the base radius and *h* is the height.

6. The distance an object falls when dropped without any propulsion is given by the formula $\boldsymbol{s = 16t^2}$ where $\boldsymbol{s}$ is the distance (in feet) it has fallen after $\boldsymbol{t}$ seconds. Because of human reaction time, using a stopwatch to time the fall of a rock has a magnitude of error of 0.2 seconds. Discuss (as above) the distance the rock has fallen if you report it fell for 3.5 seconds.

Specifications and Tolerances

The ideas of uncertainty and percent uncertainty can be placed in another context: *manufacture* rather than *measurement*. There isn't *any* process that can produce *any* item to have *exactly* the dimensions specified. The designer of a machine must understand the capabilities of the processes that will be used to produce the component parts so that he or she does not expect greater precision than is possible. Thus, the diameter of a driveshaft, for example, might be *specified* as (1.125 ± 0.003) inches, with ±0.003 inches being the *tolerance* allowed for that dimension. The value—0.003—is not measurement uncertainty; instead, it might be thought of as *manufacturing uncertainty*, the absolute value of the maximum manufacturing error that can be tolerated in the production of the driveshaft. The bracketing interval is seen to be $1.122\ in \le \boldsymbol{X_A} \le 1.128\ in$.

How tolerances should be combined mathematically to produce the maximum manufacturing error that will arise when parts are *assembled* could be a course in itself. Much depends on the *distribution* of values. For example, is a driveshaft diameter between 1.125 and 1.126 inches more likely to occur than one between 1.127 and 1.128? We will study data distributions in Chapter 6, including the *normal distribution* in §6C. Although the normal distribution frequently describes quantities that have random variation, oftentimes an understanding of a particular manufacturing process will show that the distribution of values does not resemble a normal distribution at all.