Logarithms

The purpose of this document is to provide sufficient information about logarithmic functions so that students can solve exactly certain kinds of problems in exponential growth and decay, particularly problems in compound interest.

Introduction

Generally, when one encounters a mathematical formula, it is possible to solve for any one of the variables in terms of the other(s). Consider one of the most familiar formulas—at least to students who have taken an elementary algebra course—"Distance equals Rate times Time", $D = R \cdot T$. Either one of the variables *Rate* or *Time* can be found by dividing both sides of the equation by the other variable: $\frac{D}{R} = \frac{\cancel{R} \cdot T}{\cancel{R}} = T$ and $\frac{D}{T} = \frac{R \cdot \cancel{T}}{\cancel{T}} = R$. The formula for the area *A* of a circle, in terms of its radius *r*, can be solved for radius in terms of area: $A = \pi r^2$, so $r = \sqrt{A/\pi}$. The formula for the volume *V* of a right circular cone of base radius *r* and height *h* can be solved for base radius or for height: $V = \frac{1}{3}\pi r^2 h$, so $r = \sqrt{\frac{3V}{\pi h}}$ and $h = \frac{3V}{\pi r^2}$. But how on earth can one solve a compound interest formula for the number of time periods required to reach a certain goal? Problem 45 of §4A illustrates the use of trial and error for such a problem. Isn't there a better way? That's what this material is all about, and the answer is "Yes".

Finding the Proper Exponent

One learns early to think of an exponent as a kind of "repetition count" in a multiplication problem, although a fairly elementary algebra course will take one well beyond this simplistic definition. But think about some of the early examples we learn: $2^2 = 2\cdot2 = 4$, $2^3 = 2\cdot2\cdot2 = 8$, $3^3 = 3\cdot3\cdot3 = 27$ and $3^4 = 3\cdot3\cdot3\cdot3 = 81$. Now let's give our imaginations free rein. What might $2^{2.99}$ or 3^{π} represent? Well, 2.99 is very nearly 3, so we would expect $2^{2.99}$ to be *almost* equal to 8. And $\pi \approx 3.1416$, so wouldn't 3^{π} be a number somewhat larger than 27 but much closer to 27 than to 81? What's needed is a way to figure out the exponent to which any base should be raised in order to produce whatever number is desired. The fact that the base (the *2* or *3* in our examples above) makes a difference complicates matters. For now, let's consider only one base, *10*. We will seek exponents of 10 that will yield whatever result we desire.

The first step is to do a reality check. We know that $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$ and so on. Perhaps you even remember that $10^0 = 1$ and that $10^{-1} = 0.1$. It's beginning to look as if "whatever result we desire" will take a bit of a hit—we'll have to be content with results that are *positive numbers*.

> **Definitions:** A **logarithm** (or **log** for short) is a number that represents the exponent or power to which a base must be raised to produce a certain positive result. If the base is 10, the logarithm is known as a **common logarithm**. So $\log_{10} x$ is the *power* to which *10* must be raised to obtain *x*. Here it is in symbols.

Logarithm Property #1: $10^{\log_{10} x} = x \qquad (x > 0)$ **A *logarithm is an exponent*.**

Logarithm Property #2: $\log_{10} x = y$ **<u>means</u>** $10^y = x \quad (x > 0)$ As in #1, ***the logarithm is the exponent.***

Example 1

Find $\log_{10} 100000$, $\log_{10} 10^{43}$ and $\log_{10} 0.0000001$ without using a calculator.

Solution:

We recognize 100000 as 10^5, so the answer to the question "What power of 10 gives 10^5?" is 5! Thus, $\log_{10} 100000 = 5$. Even more readily we find $\log_{10} 10^{43} = 43$. Since $0.0000001 = 10^{-7}$, $\log_{10} 0.0000001 = -7$. ***The logarithm is the exponent.***

When we consider the second part of Example 1, we see that the number 43 could have been replaced by any real number. Property #2 also leads to the same conclusion, if *x* is eliminated between the two equations. That is

Logarithm Property #3: $\log_{10} 10^y = y$ Again, ***the logarithm is the exponent.*** Note $\log_{10} 1 = 0$ and $\log_{10} 10 = 1$.

Evaluating Logarithms

Example 2

Using the **log** key on a calculator, confirm that $\log_{10} 100000 = 5$. Also find $\log_{10} 2$, $\log_{10} 50$, $\log_{10} \frac{1}{2}$ and $\log_{10} 0.0002$.

Solution:

On a TI-83+ or similar calculator, press $\boxed{\text{log}}$ 100000 $\boxed{=}$. On a simpler calculator, press 100000 $\boxed{\text{log}}$.

Example 3

How is the logarithm of 10^{75} related to the logarithms of 10^{50} and 10^{25}? Use base 10.

Solution:

The logarithm is the exponent! That means that $\log_{10} 10^{75} = 75$, $\log_{10} 10^{50} = 50$ and $\log_{10} 10^{25} = 25$. Note that $75 = 50 + 25$. In other words $\log_{10} 10^{75} = \log_{10} 10^{50} + \log_{10} 10^{25}$. The logarithm of 10^{75} is the sum of the logarithms of 10^{50} and 10^{25}. It's useful to note that a rule learned long ago for exponents tells us that 10^{75} is the *product* of 10^{50} and 10^{25}.

The relationship found in Example 2 can be generalized to give a result for any product of positive numbers. The rule for exponents that was alluded to is $10^u \cdot 10^v = 10^{u+v}$. Now let $x = 10^u$ and $y = 10^v$. So $10^{u+v} = x \cdot y$. But Property #2 lets us change each of these exponential statements to a logarithmic statement, namely that $\log_{10} x = u$, $\log_{10} y = v$ and $\log_{10} x \cdot y = u + v$. Substituting the first two of these into the third gives the *addition rule* for logarithms or

Logarithm Property #4: $\log_{10} (x \cdot y) = \log_{10} x + \log_{10} y \qquad (x > 0 \text{ and } y > 0)$

Example 4

Add the values, obtained in Example 2, for $\log_{10} 2$ to $\log_{10} 50$ confirm that the result is 2 (*i.e.,* $\log_{10} 100$).

Solution:

$0.301029995663981195213738894724493 + 1.698970004336018804786261105527551 \approx 2$

Since a power is an extension of a product, Property #4 may be extended to give a rule for the log of a power. Note that, if y were replaced by x, Property #4 would state that $\log_{10} (x \cdot x) = \log_{10} x + \log_{10} x$, which might also be written $\log_{10} x^2 = 2 \cdot \log_{10} x$, $x > 0$. Similarly, the logarithm of any power can be written as the product of the exponent and the logarithm of the base:

Logarithm Property #5: $\log_{10} a^x = x \cdot \log_{10} a \qquad (a > 0)$

Example 5

You deposit \$1000 in an account that pays an APR of 7% compounded monthly. How long will it take for your balance to reach \$2700?

Solution:

The Compound Interest Formula for Interest Paid n Times per Year (p. 210) is

$$A = P\left(1 + \frac{APR}{n}\right)^{(nY)}$$

Note that $1 + \frac{APR}{n} = 1 + \frac{0.07}{12} \approx 1.00583$, so we need to solve for Y in the equation $2700 = 1000 \cdot 1.00583^{12Y}$. With your new-found knowledge of logarithms, you can do so! Using Properties 4 and 5 gives

$$\log_{10} 2700 = \log_{10} 1000 + 12Y \cdot \log_{10} 1.00583$$

$$\frac{\log_{10} 2700 - \log_{10} 1000}{\log_{10} 1.00583} = 12Y$$

$$Y = \tfrac{1}{12} \cdot \frac{\log_{10} 2700 - 3}{\log_{10} 1.00583} \approx 14.24$$

This is almost 14¼ years. Your answer must be rounded upward to the next twelfth of a year, so you conclude that it will take 14 years, 3 months to reach your goal.

Logarithms with Base e

At the outset we mentioned that logarithms are exponents and that their values depend on the base to be raised to the power represented by that exponent. We then limited our discussion to common logarithms, those with base 10. Another popular logarithmic base is the number *e* introduced in the discussion of continuous compounding. Such logarithms are called ***natural logarithms***, and most calculators have a key to evaluate them as well as common logs. Usually the key is labeled $\boxed{\ln}$ and, in mathematics books, a special notation is used, ***ln x*** represents the natural logarithm of *x*. All of the properties of logarithms can be restated for natural logarithms.

Logarithm Property #1: $e^{\ln x} = x$ $(x > 0)$ **A *logarithm is an exponent*.**

Logarithm Property #2: $\ln x = y$ **means** $e^y = x$ $(x > 0)$ As in #1, ***the logarithm is the exponent.***

Logarithm Property #3: $\ln e^y = y$ Again, ***the logarithm is the exponent.*** Note $\ln 1 = 0$ and $\ln e = 1$.

Logarithm Property #4: $\ln (x \cdot y) = \ln x + \ln y$ $(x > 0$ and $y > 0)$

Logarithm Property #5: $\ln a^x = x \cdot \ln a$ $(a > 0)$

Example 6

A friend tells you that he has found a bank where he can double his money in 8.25 years with continuous compounding. What *APR* will he be receiving?

Solution:

The Compound Interest Formula for Continuous Compounding (p. 213) is

$$A = P \times e^{(APR \times Y)}$$

Note that, regardless of the principal *P*, the accumulated balance after 8.25 years is twice *P*. That is, $A = 2 \cdot P$. We need to solve for *APR* in the equation $2 = e^{8.25 \times APR}$. As in Example 5, we will take the logarithm of each side of the equation (*natural logarithm* this time) and this time use Property 3.

$$\ln 2 = 8.25 \times APR$$

$$APR = \frac{\ln 2}{8.25} \approx 0.084 = 8.4\%$$

The interest rate is 8.4%.

Example 7

You deposit \$2200 in an account that pays an *APR* of 5¾% compounded continuously. How long will it take for your balance to reach \$4000?

Solution:

The same formula is used as in Example 6, but this time the *APR* is known and the time unknown. The equation to be solved is $4000 = 2200 \times e^{(0.0575 \times Y)}$ and we must solve this equation for *Y*. To do so we take the natural logarathm of each side.

$$\ln 4000 = \ln 2200 + 0.0575 \times Y$$

$$Y = \frac{\ln 4000 - \ln 2200}{0.0575} \approx 10.4$$

The time required is 10.4 years.

Exercises

1. Given that $\log_{10} 2 \approx 0.301$, find an approximate value of $\log_{10} 64$ without using a calculator. (Hint: what power of 2 is 64?)
2. Given that $\log_{10} 2 \approx 0.301$ and $\log_{10} 3 \approx 0.477$, find, without using a calculator, approximate values of (a) 3/2 (b) 2/3 (c) 18 (d) 24.
3. How long will it take your money to triple at an *APR* of 7¾% compounded quarterly?
4. How long will it take your money to grow by 75% at an *APR* of 6% compounded monthly?
5. You deposit \$1500 in an account that pays an *APR* of 8.5% compounded semi-annually. How long will it take for your balance to reach \$40,000?
6. How long will it take your money to grow by 150% at an *APR* of 6¼% compounded continuously?
7. You see an advertisement for a savings account that promises to double your money in nine years. You decide to deposit \$10,000. How long will it take for the balance in your account to reach \$40,000? (**Quick! Don't use a formula! This should take only a few seconds.)**
8. How long will it take your money to triple at an *APR* of 7¾% compounded continuously?
9. What is the *APR* if your money triples in 14 years with continuous compounding?
10. You deposit \$1500 in an account that pays an *APR* of 8.5% compounded continuously. How long will it take for your balance to reach \$40,000?