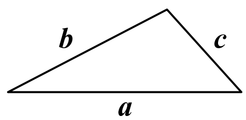
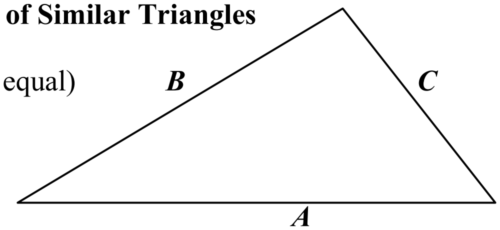
Trigonometry

Law of Similar Triangles

Given two *similar* triangles:
(i.e., corresponding angles are equal)

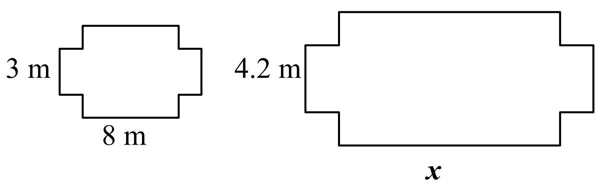


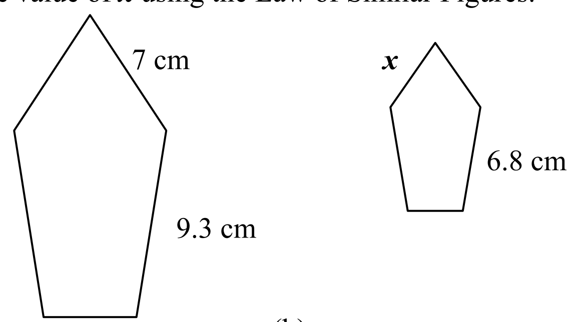


Then: $\frac{a}{b}=\frac{A}{B}$; $\frac{b}{a}=\frac{B}{A}$; $\frac{a}{c}=\frac{A}{C}$; $\frac{c}{a}=\frac{C}{A}$; $\frac{b}{c}=\frac{B}{C}$; $\frac{c}{b}=\frac{C}{B}$

Example 1

In each part below the figures are similar. Find the value of x using the Law of Similar Figures.





Solution:

(a) Here we'll take the ratios of bottom edge to side edge in each figure:

$$\frac{x}{4.2}=\frac{8}{3}$$
$$3x=4.2(8)=33.6$$
$$x=\frac{33.6}{3}=11.2$$

So, x = 11.2 meters.

(b) Now we'll take the ratios of top edge to bottom side edge in each figure:

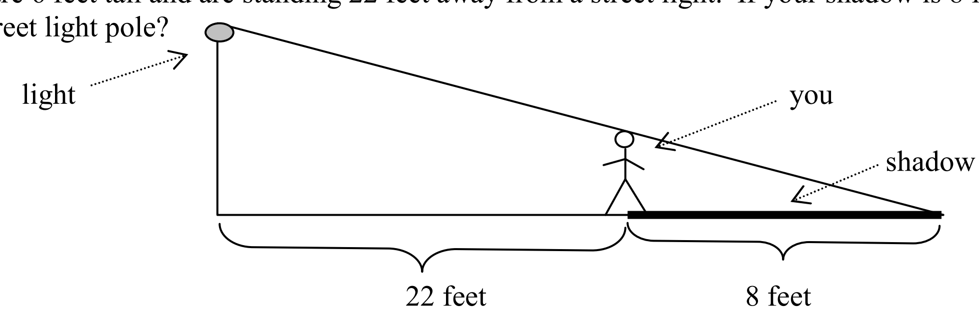
$$\frac{x}{6.8}=\frac{7}{9.3}$$
$$9.3x=6.8(7)=47.6$$
$$x=\frac{47.6}{9.3}=\frac{476}{93}\approx 5.12$$

So, x is 476/93 cm or roughly 5.12 cm.

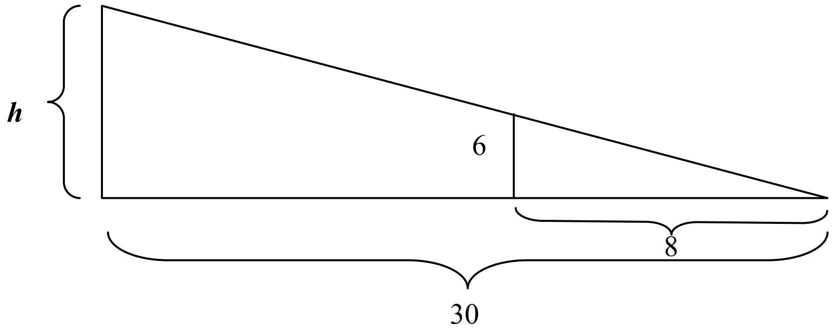
Example 2

Suppose you are 6 feet tall and are standing 22 feet away from a street light. If your shadow is 8 feet long, how tall is the street light pole?

Solution:



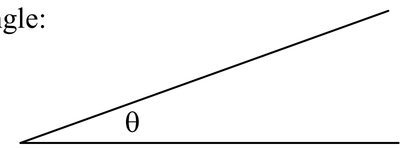
This can be viewed as similar triangles as shown below where ***h*** is the height of the light pole.



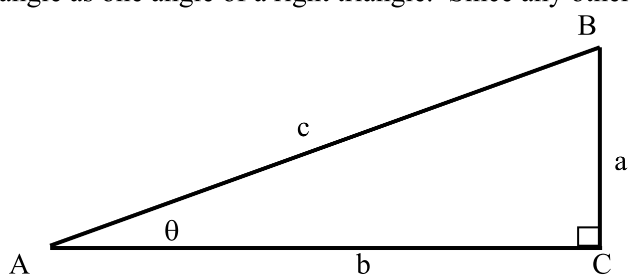
Using the Law of Similar Triangles we have: $\frac{\mathbf{h}}{\mathbf{30}} = \frac{\mathbf{6}}{\mathbf{8}}$. Solving for ***h*** gives a height of 22½ feet.

The process of the last example (where you find the length of one side of a triangle by using the Law of Similar Triangles) is one form of a process called **triangulation**.

The last example also leads us into an area of mathematics called trigonometry. To see how that comes about, let's begin with an acute angle:



Now let's use this angle as one angle of a right triangle. Since any other right triangle having this angle as one of its acute angles will be similar to the one shown above, the ratios of corresponding sides would also be equal. Thus, in a very real sense these ratios are determined by the size of the angle θ with which we started, and the values of each of these ratios have names. Using the notation in the above figure:

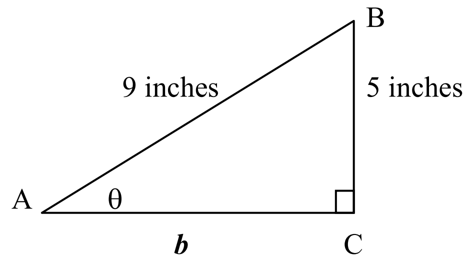


name	***symbol***	***value in words***	***value in symbols***
sine	***sin(θ)***	$\frac{\text{opposite}}{\text{hypotenuse}}$	$\frac{\boldsymbol{a}}{\boldsymbol{c}}$
cosine	***cos(θ)***	$\frac{\text{adjacent}}{\text{hypotenuse}}$	$\frac{\boldsymbol{b}}{\boldsymbol{c}}$
tangent	***tan(θ)***	$\frac{\text{opposite}}{\text{adjacent}}$	$\frac{\boldsymbol{a}}{\boldsymbol{b}}$

Example 3

Find the following based on the figure on the right.

(a) b (b) $sin(\theta)$ (c) $cos(\theta)$ (d) $tan(\theta)$



Solution:

(a) Using the Pythagorean Theorem we have:

$$9^2 = 5^2 + b^2$$
$$b^2 = 81 - 25 = 56$$
$$b = \sqrt{56} \approx 7.48 \text{ inches}$$

(b) $sin(\theta) = \dfrac{opposite}{hypotenuse} = \dfrac{5 \text{ inches}}{9 \text{ inches}} \approx \mathbf{0.5556}$

(c) $cos(\theta) = \dfrac{adjacent}{hypotenuse} = \dfrac{\sqrt{56} \text{ inches}}{9 \text{ inches}} \approx \mathbf{0.8315}$

(d) $tan(\theta) = \dfrac{opposite}{adjacent} = \dfrac{5 \text{ inches}}{\sqrt{56} \text{ inches}} \approx \mathbf{0.6682}$

You may have already noticed that we have only looked at three of the six ratios determined by the sides of a triangle. The other ratios (e.g., $\frac{hypotenuse}{opposite}$) also have trigonometric names which you may have heard of: ***cosecant***, ***secant*** and ***cotangent***. We don't need those for our work here, so we won't define them.

We can gather some important information about these functions based on the definitions and the above example.

- *The values of the trigonometric functions are unitless*. Notice in the example above that "inches" appears in both numerator and denominator and hence cancels out. (This, however, presupposes that we use the same units for measuring each side of the triangle – in fact, this is *required* if you are going to calculate any of the trigonometric functions using those measurements.
- Since both the sine and cosine functions use the hypotenuse in the denominator, and since we know the hypotenuse is the longest side of a right triangle, we have that *for any acute angle θ the following inequalities hold:* $0 < sin(\theta) < 1$ *and* $0 < cos(\theta) < 1$.
- If we go back to the definitions using ***a, b*** and ***c***, we find that

$$\frac{sin(\theta)}{cos(\theta)} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{c} \cdot \frac{c}{b} = \frac{a}{b} = tan(\theta).$$

Thus, $tan(\theta) = \dfrac{sin(\theta)}{cos(\theta)}$.

- From the Pythagorean Theorem we have $a^2 + b^2 = c^2$. If we divide all three terms by c^2, we get the following:

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$$
$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$
$$\left(sin(\theta)\right)^2 + \left(cos(\theta)\right)^2 = 1$$

This is one of three identities (we don't need the other two) which are called the *Pythagorean Identities*. We use a special notation for raising trigonometric functions to positive powers: $sin^2(\theta) = \left(sin(\theta)\right)^2$. So we have:

$$sin^2(\theta) + cos^2(\theta) = 1$$

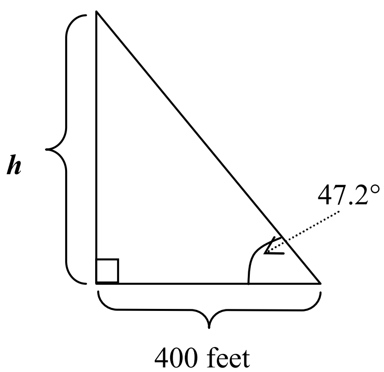
Since the values of the trigonometric functions are determined by the angle, not the side lengths of any particular triangle in which it sits, there are a variety of high powered mathematical ways of calculating the trigonometric values of any such angle. The results are incorporated into our calculators. To find the sine of, say, 40°, we just punch it into the calculator (in degree mode): ***sin(40) ≈ 0.6428***. (As a general rule, we'll round these values to 4 decimal places.)

Example 4

From a point 400 feet on the ground away from the base of a building you look up to the top of the building. Using an angle measuring device you find that the *angle of elevation* (angle from the horizontal to the line of sight) is 47.2°. How tall is the building?

Solution:

The sketch below models this situation, where ***h*** represents the unknown height:



With respect to the given angle, the length ***h*** in feet is the opposite side of the triangle and the 400 feet base line is the adjacent side. Thus, we use the tangent function:

$$\tan(47.2^\circ) = \frac{h \text{ feet}}{400 \text{ feet}}$$
$$h = 400 \cdot \tan(47.2^\circ)$$
$$h \approx 400 \times 1.0799 \approx 432 \text{ feet}$$

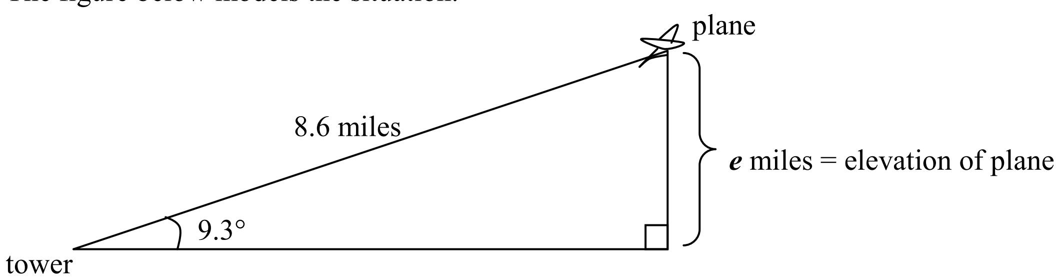
So, we see the building is roughly 432 feet tall.

Example 5

An air traffic controller spots a plane having an angle of elevation of 9.3°. Radar indicates the straight line distance from the tower to the plane is 8.6 miles. What is the elevation of the plane?

Solution:

The figure below models the situation.

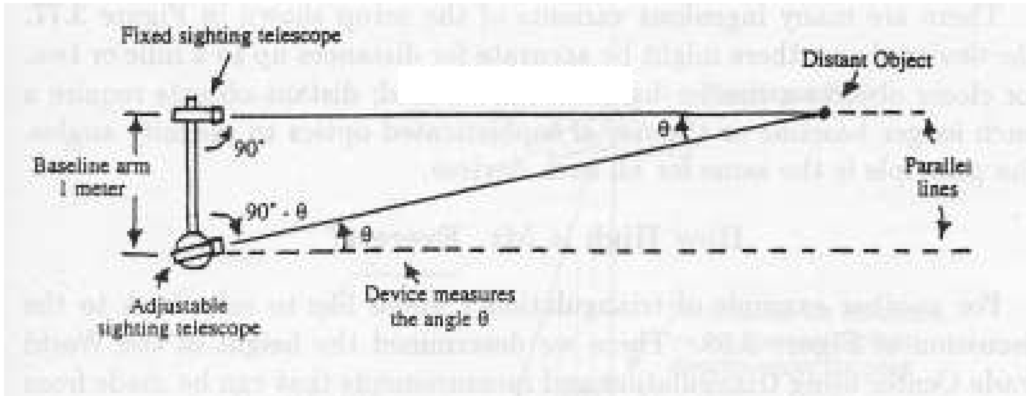


Here, with respect to the given angle, we know the hypotenuse and want to find the opposite side. That indicates the sine function is convenient to use.

$$\sin(9.3^\circ) = \frac{e \text{ miles}}{8.6 \text{ miles}} = \frac{e}{8.6}$$
$$e = 8.6 \cdot \sin(9.3^\circ) \approx 8.6 \times 0.1616 \approx 1.4 \text{ miles}$$

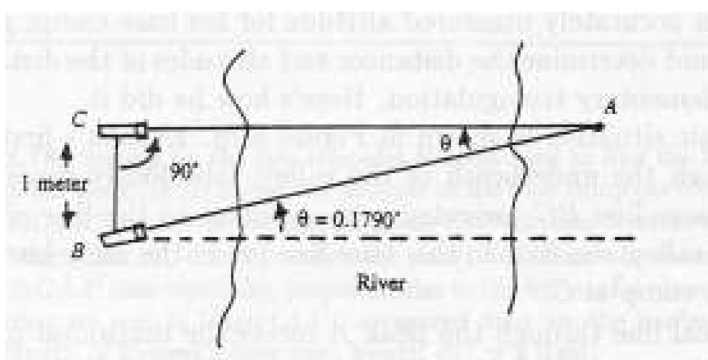
A *range finder* is a surveying device which allows us to find the distance to distant objects that we can see but can't necessarily easily get to. It consists of two telescopes mounted on opposite ends of a 1 meter rod. One telescope is mounted perpendicular to the rod, and the other telescope is mounted on a swivel pivot on a platform that has an angle grid on it. See the diagram on the next page. The method is to sight the distant object in the fixed telescope, then rotate the movable one to also sight the object. The angle θ as shown in the diagram is then read off of the platform.

A Range Finder



Example 6

In the figure below a range finder has been set up on the bluffs of Manhattan Island, near the George Washington Bridge. Point A is on the Palisades across the river in New Jersey. If the observed angle is θ = 0.1790°, find the distance across the river.



Solution:

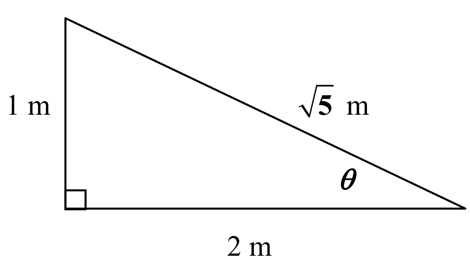
We're after the length AC here and we know the length BC is 1 meter. From the diagram we can see that we can use the tangent of the angle to find the length AC, which we'll call ***w*** for the width of the river.

$$\tan(0.1790°) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{length } BC}{\text{length } AC} = \frac{1\text{ meter}}{w}$$

$$w = \frac{1}{\tan(0.1790°)} \approx \frac{1}{0.00312415} \approx 320 \text{ meters}$$

We used more than 4 decimal places in the value for the tangent here since the value was so small; this is typical when working with a range finder.

There are often times when you know the lengths of 2 or 3 sides of a right triangle and need to find the size of the acute interior angles. For example, suppose we have the following right triangle, and we wish to



know the size of angle θ. We certainly know the $sin(\theta) = \frac{1}{\sqrt{5}}$, $cos(\theta) = \frac{2}{\sqrt{5}}$ and $tan(\theta) = \frac{1}{2}$, but we want to know about θ. Again, thanks to some fancy mathematics the angle θ satisfying the above equalities can be determined. The mathematics for doing that have been programmed into your calculator and are accessible by using any of the keys marked sin^{-1}, cos^{-1} or tan^{-1} (which you get by using the 2nd key and the ***sin***, ***cos*** or ***tan*** key, respectively. These are the **inverse trigonometric functions**. They act on ratios and produce angles.

name 1	*name* 2	*symbol* 1	*symbol* 2
inverse sine	arcsine	sin^{-1}	***arcsin***
inverse cosine	arccosine	cos^{-1}	***arccos***
inverse tangent	arctangent	tan^{-1}	***arctan***

We enter the value of the corresponding trigonometric function as the argument of the inverse trigonometric function. So, we enter on the calculator $sin^{-1}\left(1/\sqrt{\ }(5)\right)$ and we get roughly 26.57°. Similarly, if we enter $cos^{-1}\left(2/\sqrt{\ }(5)\right)$, we get roughly 26.57°. And just to show you we are consistent here, if we enter $tan^{-1}\left(1/2\right)$, we again get roughly 26.57°.

Be very careful here, though, and do not confuse the special symbol $sin^{-1}\, x$ for the inverse sine function which acts on ratios to produce angles and the symbol $\left(sin(x)\right)^{-1} = \frac{1}{sin(x)}$ which is the reciprocal of the sine function which acts on angles to produce ratios: $sin^{-1}(x) \neq \frac{1}{sin(x)}$.

Example 7

(a) If you were told the tangent of an angle is 3.5, what is the angle?
(b) If you were told the cosine of an angle is 3.5, what would you say?

Solution:

(a) If we let θ be the angle, then we are told $tan(\theta) = 3.5$. So, $\theta = tan^{-1}(3.5) \approx 74.05°$

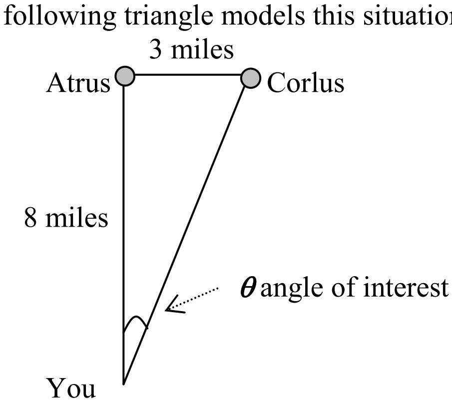
(b) If we let θ be the angle, then we are told $cos(\theta) = 3.5$. But we then fall on the floor and die laughing, since we know that $0 < cos(\theta) < 1$. So, this is impossible.

Example 8

According to a sea navigation map, the island of Atrus is 8 miles due North of your current position. The island of Corlus is 3 miles due East of Atrus. If you want to head toward Corlus, what heading should you use?

Solution:

The following triangle models this situation.



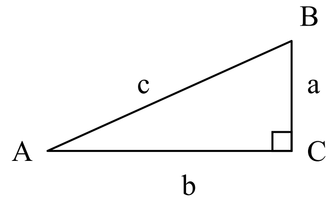
With respect to θ we know the lengths of the adjacent and opposite sides of the angle. So we use the inverse tangent function:

$$\theta = tan^{-1}\left(\frac{3}{8}\right) \approx 20.556°.$$

So, our heading should be 20.556° East of North. (This would be written N20.556°E in nautical terms.)

Exercises

In the following, the *standard ABC right triangle* is any right triangle labeled as shown here.



1. In each part you are given some information about an acute angle θ. Find the requested value using the bullet point information on page 4 of the Hand Out.

(a) $\boldsymbol{\sin\theta \approx 0.42}$; find $\boldsymbol{\cos\theta}$ {0.9075}

(b) $\boldsymbol{\cos\theta \approx 0.885}$; find $\boldsymbol{\sin\theta}$ {0.4656}

(c) $\boldsymbol{\sin\theta \approx 0.7771,\ \cos\theta \approx 0.6293}$; find $\boldsymbol{\tan\theta}$ {1.2349}

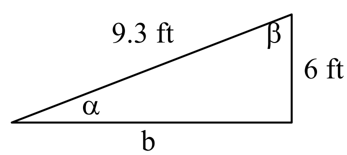
(d) $\boldsymbol{\sin\theta \approx 2.131}$; find $\boldsymbol{\cos\theta}$ {not possible}

2. (a) What is the measure of each angle of an equilateral triangle?
(b) In an equilateral triangle, draw a line segment from one vertex to the midpoint of the opposite side. Explain why the two triangles formed are right triangles and why they are identical. (The fancy word for two identical geometric figures is **congruent**.)
(c) Assign lengths to all three sides of one of the two right triangles formed in part (b). Use these lengths to find the sine, cosine and tangent of the two acute angles. SHOW WORK
(d) Compare your results with a classmate. Did you get the same result? Why?

3. We cannot create right triangles with interior angles of measure 0° nor can we create right triangles with two interior angles each measuring 90°. Hence, sine and cosine of 0° and 90° cannot be defined using the right triangle definitions we gave in the Hand Out. However, argue why it might make sense, based on what we've done, to assign the following values:

$$\boldsymbol{\sin(0^\circ) = 0 \qquad \cos(0^\circ) = 1}$$
$$\boldsymbol{\sin(90^\circ) = 1 \qquad \cos(90^\circ) = 0}$$

4. Using the following triangle, find the requested information.



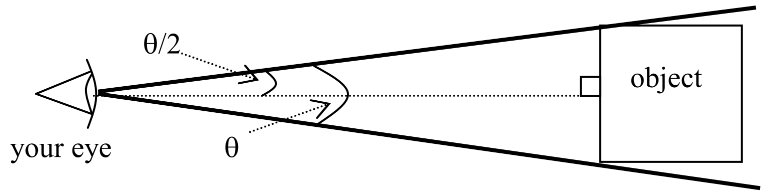
(a) b
(b) sin α
(c) cos β
(d) cos α
(e) sin β
(f) tan α

5. In problem 4 above, notice your answers to parts (b) and (c) along with your answers to parts (d) and (e). By looking at an arbitrary right triangle and the definition of the sine and cosine functions, make (and **prove**) a generalization of your observation.

6. In the standard ABC right triangle:
(a) if a = 12 yards and ∠A = 21.4°, find b and c,
(b) if b = 6 cm and c = 13 cm, find ∠A and ∠B,
(c) if b = 43 m and ∠A = 35°, find a and c,
(d) if a = 52 feet and c = 40 feet, find ∠A and ∠B,
(e) if c = 36 miles and ∠A = 23°, find a and b,
(f) if a = 28 in and b = 16 in, find ∠A and ∠B.

7. Two identical fourteen-foot ladders are bolted together at the top and their bases are separated so that the ladders form an inverted "V" 12' tall. (a) What angle does each ladder make with the ground? (b) How far apart would the bases be if that angle were increased to 65°?

The **angular diameter** of an object in our line of sight is the angle θ in the diagram below. Note also the angle θ/2 which is one of the acute angles in each of the right triangles formed.



8. (a) If a quarter is held a yard away from your eye, what is its angular diameter? {≈1.52°}
(b) On a certain date the distance between Jupiter and the earth was 7.23×10^{8} km. Jupiter has a diameter of 1.428×10^{5} km. What was the angular diameter of Jupiter from the Earth on that date? {≈0.011°}

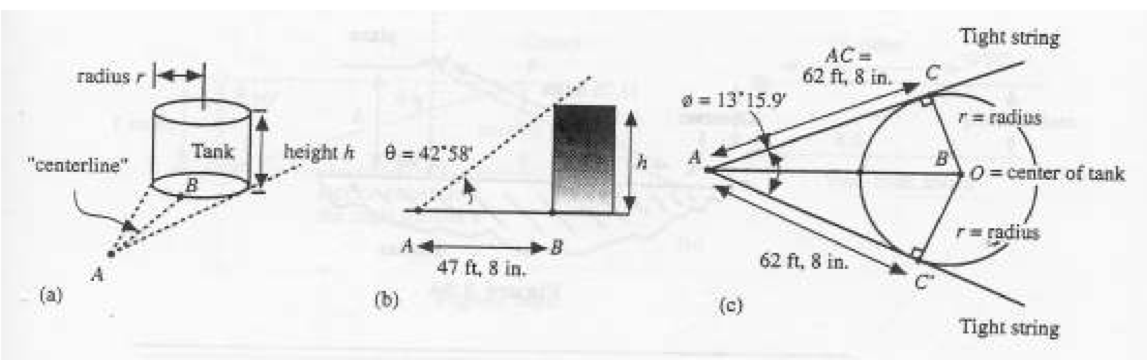
A **light year** is not a time unit, rather it is a distance – 1 **light year** is the distance light travels in a vacuum in one year. It is roughly 5.8798×10^{12} miles. One **Astronomical Unit** (1 **AU**) is the mean distance from the Earth to the Sun and is roughly 9.296×10^{7} miles.

9. A binary star system consists of two stars that are close to each other and in fact rotate about each other (i.e., rotate about a common turn point). A relatively close binary star system is the Alpha Centauri system which is about 4.3 light years away. Various telescopes have found the angular diameter of the pair (which is essentially the angle between them as seen from the Earth) is 0.0206389°. how far apart are these stars (a) in light years? (b) in AU's?

10. Venus, on a certain day, was about 0.1696 AU from the Earth. Its apparent angular diameter as seen from the Earth was about 0.013667°. Approximate the diameter of Venus in miles. {3760 miles}

11. A kite, anchored at ground level, is flying in a steady stiff breeze. The string to the kite is 115 feet long, and the angle the string makes with the ground is 18°. Find the vertical height of the kite.

12. To find the volume of a large cylindrical tank measurements are made as shown in the figures below. Find the volume of this tank. (Note: 42°58' ≈ 42.9667°, 13°15.9' = 13.265°.) {7406 cubic feet}



13. An air traffic controller on the ground spots a plane with angle of elevation 32.05°. Her radar tells her that the direct line distance to the plane is 11.76 miles. If it is high noon and the plane's shadow is directly beneath it, find the distance between the shadow of the plane and the controller, assuming the ground is flat.

14. If a flagpole casts a 29 foot shadow and a 5 foot 3 inch tall person casts a shadow that is 3 feet 6 inches long, estimate the height of the pole. {43.5 feet}

15. An observer on a ship uses a range finder to determine the distance to a light house. The range finder gives an angle of 0.045°. How far is the ship from the light house in miles?

16. Two wooden planks are hinged together end-to-end and set up to form an inverted "V". Unlike the ladders in problem 7, however, these planks are not the same length; one is 8 ft long, the other 2 ft long. When the 2-ft plank makes an angle of 10° with the ground, (a) how high is the highest point on the "V"?, (b) what angle does the 8-ft plank make with the ground?, (c) how far apart are the unattached ends of the planks?

17. Repeat problem 16, with the angle between the 2-ft plank and the ground being 30°.