Completing the Square and the Quadratic Formula

(1) We now have three of the five methods for solving quadratic equations – can you name and demonstrate each of them? We'll now take on the task of learning the last two methods.

(2) The first of these relies on observing how simple it was to solve quadratic equations of the form $(x-4)^2 = 7$ by using the Perfect Square Rule. The left side of this equation is the square of a binomial difference and would look like $x^2 - 8x + 16$ if expanded. We would recognize this as a perfect square trinomial based on the hundreds of factoring problems we've done. So, had the equation been given as $x^2 - 8x + 16 = 7$, we could factor the left side and get $(x-4)^2 = 7$ and then use the Perfect Square Rule. But what if we were given $x^2 - 8x + 9 = 0$? The observant student should see how I got this – subtract **7** from both sides of $x^2 - 8x + 16 = 7$. Our quest is to do the reverse of this – start with $x^2 - 8x + 9 = 0$ and discover how to manipulate it into a perfect square trinomial equaling a number.

(3) The method is called **completing the square**, and it relies on understanding the form of a perfect square trinomial a bit better:

$$(x+a)^2 = x^2 + 2ax + a^2$$

Notice the linear coefficient, $2a$. Taking one half of it, $\frac{1}{2}(2a) = a$, gives the object which is added to x in the binomial, $x + a$, and whose square, a^2, appears as the constant in the trinomial. So that is how perfect square trinomials are formed: **the constant is the square of $\frac{1}{2}$ of the linear coefficient**.

(4) We use the above concept to create a perfect square trinomial by the technique called **completing the square**.

Example 1: Solve $x^2 + 6x - 2 = 0$

Solution: In order to create the proper constant to have this be a perfect square trinomial, we first make room for it by, in this case, adding **2** to both sides.

$x^2 + 6x \quad = 2$	Make room for new constant
$\frac{1}{2}(6) = 3$	Take $\frac{1}{2}$ the linear coefficient
	The square of **3** is the constant we need to make a perfect square trinomial, $x^2 + 6x + 9$. But we can't just add **9** to one side, we have to add it to BOTH sides
$x^2 + 6x + 9 = 2 + 9$	Simplify
$(x+3)^2 = 11$	Use Perfect Square Rule
$x + 3 = \pm\sqrt{11}$	Subtract 3 from both sides
$x = -3 \pm \sqrt{11}$	

Example 2: Solve $x^2 - 5x + 3 = 0$
Solution:

$x^2 - 5x + 3 = 0$ — Make room for new constant

$x^2 - 5x \quad = -3$ — Take half the linear coefficient

$\frac{1}{2}(-5) = -\frac{5}{2}$ — The square of this is the new constant which we add to BOTH sides

$$x^2 - 5x + \left(-\frac{5}{2}\right)^2 = -3 + \left(-\frac{5}{2}\right)^2$$

$\left(x - \frac{5}{2}\right)^2 = -3 + \frac{25}{4} = \frac{-12+25}{4} = \frac{13}{4}$ — Use Perfect Square Rule

$x - \frac{5}{2} = \pm\sqrt{\frac{13}{4}} = \pm\frac{\sqrt{13}}{2}$ — Add $\frac{5}{2}$ to both sides

$$x = \frac{5}{2} \pm \frac{\sqrt{13}}{2} = \frac{5 \pm \sqrt{13}}{2}$$

(5) **ZZZZZ** This method ONLY WORKS ON MONIC QUADRATICS – i.e., where the leading coefficient is **1**. If it isn't **1** in what you start with, then you must divide both sides of the equation by the leading coefficient to make monic.

Example 3: Solve $4x^2 - 8x - 3 = 0$
Solution:

$4x^2 - 8x - 3 = 0$ — Make monic by dividing by **4**

$x^2 - 2x - \frac{3}{4} = 0$ — Create constant space

$x^2 - 2x \quad = \frac{3}{4}$ — $\frac{1}{2}(-2) = -1$, whose square is added to BOTH sides

$$x^2 - 2x + (-1)^2 = \frac{3}{4} + (-1)^2$$

$(x-1)^2 = \frac{3}{4} + 1 = \frac{3+4}{4} = \frac{7}{4}$ — Use Perfect Square Rule

$x - 1 = \pm\sqrt{\frac{7}{4}} = \pm\frac{\sqrt{7}}{2}$ — Add **1** to both sides

$$x = 1 \pm \frac{\sqrt{7}}{2} = \frac{2 \pm \sqrt{7}}{2}$$

(6) We can apply this method to an arbitrary quadratic equation in standard form. The end result is a formula for x in terms of the coefficients a, b and c. It is known as the **Quadratic Formula**.

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

are the solutions to the equation

$ax^2 + bx + c = 0,$ where $a \neq 0$

(7) Let's concentrate a bit on the a, b and c in the standard form of the quadratic equation.

When the quadratic equation is in standard form

$$ax^2 + bx + c = 0$$

a is the coefficient of the quadratic term,
b is the coefficient of the linear term, and
c is the constant (object with no x in it).

This statement highlights something more than just the definitions of a, b and c. It says there can be ONLY ONE term containing x^2, there can be ONLY ONE term containing x, and the constant is the entire collection of terms which do not contain x. For example, consider $mx^2 + 4x = 3 - mx$. This is quadratic in x so we create an equation to $\mathbf{0}$: $mx^2 + 4x + mx - 3 = 0$. This is NOT in standard form since there are two linear terms, $4x$ and mx. We can ONLY have ONE linear term in order to use the Quadratic Formula, so we create a single linear term by distributing an x out of these two terms: $mx^2 + (4+m)x - 3 = 0$. Now we can identify a, b and c: $a = m,\ b = 4 + m,$ and $c = -3$. Thus, we have

$$x = \frac{-(4+m) \pm \sqrt{(4+m)^2 - 4(m)(-3)}}{2m} = \frac{-(4+m) \pm \sqrt{16 + 8m + m^2 + 12m}}{2m}$$

$$= \frac{-(4+m) \pm \sqrt{m^2 + 20m + 16}}{2m}$$

Here's another example: Solve for x in $(x-1)^2 + y^2 = 4x$. Notice first that we can't isolate the $(x-1)^2$ as $4x - y^2$ and use the Perfect Square Rule since this would place an x in the answer to a problem where we want to solve for x. So we go back to basics and multiply it out:

$(x-1)^2 + y^2 = 4x$	Remove grouping symbols
$x^2 - 2x + 1 + y^2 = 4x$	Quadratic in $x \Rightarrow$ try for standard form
$x^2 - 6x + 1 + y^2 = 0$	OK: $a = 1,\ b = -6,\ c = 1 + y^2$ That's right, remember c is made up of all terms not containing x

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(1+y^2)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36 - 4 - 4y^2}}{2} = \frac{6 \pm \sqrt{32 - 4y^2}}{2} = \frac{6 \pm \sqrt{4(8 - y^2)}}{2}$$

$$= \frac{6 \pm 2\sqrt{8 - y^2}}{2} = \frac{2\left(3 \pm \sqrt{8 - y^2}\right)}{2}$$

$$x = 3 \pm \sqrt{8 - y^2}$$