Factoring

(1) **GCF factoring**. Factoring out the Greatest Common Factor is, as we know, simply an application of the distributive law: $ab+ac = a(b+c)$. We factor out the largest numerical divisor of all the coefficients (the GCF of the coefficients) and the least power on any common variable – remembering that we end up with the same number of terms inside the parentheses as we had in the original expression. For example, $6x^3 + 3x^2 - 12x^4$ factors into $3x^2(2x + 1 - 4x^2)$. GCF factoring is the most primitive factoring and the easiest. _ALWAYS do any GCF factoring FIRST_ in any problem where factoring is needed.

(2) **Special Products factoring**.

(a) **Difference of squares**. The difference of two squares, $a^2 - b^2$, factors into a product of binomial conjugates: "the first plus the second times, the first minus the second" $a^2 - b^2 = (a+b)(a-b)$. For example, $16m^4 - 9t^{10} = (4m^2)^2 - (3t^5)^2 = (4m^2 + 3t^5)(4m^2 - 3t^5)$. The sum of two squares, $a^2 + b^2$, DOES NOT FACTOR knowing what we know so far. In general, if an expression does not factor, it is called **prime**.

(b) **Perfect Square Trinomials**. Perfect square trinomials are the result of squaring binomial sums and differences. Each of the expressions to the right of the "=" below is a perfect square trinomial.

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad\qquad (a-b)^2 = a^2 - 2ab + b^2$$

Hence, if we have a trinomial with the squares of two objects on the "outside" and either twice their product or the negation of twice their product "in the middle," we have a perfect square trinomial:

$$a^2 + 2ab + b^2 = (a+b)^2 \qquad\qquad a^2 - 2ab + b^2 = (a-b)^2$$

For example, $x^2 + 6x + 9 = (x+3)^2$, $x^2 - 10x + 25 = (x-5)^2$ and more complicated, but identical in form, we have $4x^2 - 24xy + 9y^2 = (2x - 3y)^2$.

(3) **Group and Hope**. This is a very specialized method that can be tried if you have four terms with no common factor, such as $x^2 - 2xb + 5x - 10b$. The idea is to group the terms into two sets of two, factor each group and hope something pops up that is useful. We'll look at the three different ways of grouping this expression and see what happens:

(i) $x^2 - 2xb + 5x - 10b = (x^2 - 2xb) + (5x - 10b)$ grouping terms 1&2 and 3&4

$= x(x - 2b) + 5(x - 2b)$ factor each group and magic!

$= (x+5)(x-2b)$ factor out $(x - 2b)$

(ii) $x^2 - 2xb + 5x - 10b = (x^2 + 5x) + (-2xb - 10b)$ grouping terms 1&3 and 2&4

$= x(x+5) - 2b(x+5)$ factor each group and magic again!

$= (x - 2b)(x+5)$ factor out $(x+5)$

(iii) $x^2 - 2xb + 5x - 10b = (x^2 - 10b) + (-2xb + 5x)$ grouping terms 1&4 and 2&3

$= (x^2 - 10b) + x(-2b + 5)$ yuk, no luck here

If the Group and Hope method works, you will be able to factor the expression in at least two of the three ways you can group its terms. So, if one grouping doesn't work, try another; if that also doesn't work, then this method fails.

(4) **Sum and Difference of Cubes**.

(a) $A^3 + B^3 = (A+B)(A^2 - AB + B^2)$

(b) $A^3 - B^3 = (A-B)(A^2 + AB + B^2)$

(5) Finally, we come to the fun one – **arbitrary trinomials**. Yes, we are talking about things like $x^2 + 5x + 6$ which are quadratic trinomials which are not perfect squares. In order to factor such expressions, we have to fully understand how they are formed through what you probably call FOIL but I call LipopR. Examine carefully the form of the three products below which produce MONIC quadratic trinomials. (Recall, **monic** means leading coefficient is 1.)

For a and b POSITIVE:

(A) $(x+a)(x+b) = x^2 + (a+b)x + ab$

(B) $(x-a)(x-b) = x^2 - (a+b)x + ab$

(C) $(x+a)(x-b) = x^2 + (a-b)x - ab$

Observe: When the trinomial's CONSTANT is POSITIVE (cases (A) and (B)), the two binomials have the same middle sign. In that case, if the linear coefficient is positive (case (A)), the binomial middle signs are " + "; and if the linear coefficient is negative (case(B)), the binomial middle signs are " – ". The only time the binomial middle signs are opposite of each other is when the trinomial's CONSTANT is NEGATIVE (case (C)). Thus, we know that if the following trinomials factor, then the factorization has the stated form.

$x^2 + 5x + 6 = (x+a)(x+b)$ $\qquad$ $x^2 - 7x + 6 = (x-a)(x-b)$ $\qquad$ $x^2 + x - 6 = (x+a)(x-b)$

We also observe that the trinomial's constant (ab)is the product of the two binomial constants (a and b). So we just have to think of all the ways the trinomial's constant factors and check the resulting ipop (inside product + outside product) to find the factorization.

Some examples.

$x^2 + 5x + 6 = (x+a)(x+b)$	$ab = 6$ so we have $1 \cdot 6$ or $2 \cdot 3$
	but ipop is $5x$ so $a + b = 5$
	try them: $1 + 6 \neq 5,\ 2 + 3 = 5$
$= (x+2)(x+3)$	done when found $2 + 3 = 5$

$x^2 - 7x + 6 = (x-a)(x-b)$	$ab = 6$ so again $1 \cdot 6$ or $2 \cdot 3$
	ipop is $-7x$ so $a + b = 7$
	try them: $1 + 6 = 7,\ 2 + 3 \neq 7$
$= (x-1)(x-6)$	done after $1 + 6 = 7$, but I showed the other for completeness

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$x^2 + x - 6 = (x+a)(x-b)$	$ab = 6$ so again $1 \cdot 6$ or $2 \cdot 3$
	ipop is $+1x$ so $a - b = +1$
	try them: $1 - 6 \neq 1,\ 6 - 1 \neq 1,\ 2 - 3 \neq 1$
	$3 - 2 = 1$, finally
$= (x+3)(x-2)$	

$$x^2 - 6x + 8 = (x-a)(x-b) \qquad ab = 8,\ a+b = 6\ \Rightarrow a = 2,\ b = 4$$
$$= (x-2)(x-4)$$

$$x^2 - 2x - 15 = (x+a)(x-b) \qquad ab = 15,\ a-b = -2\ \Rightarrow a = 3,\ b = 5$$
$$= (x+3)(x-5)$$

(6) If you have this down well, the jump to non-monic quadratic trinomials is not bad. There are two main strategies one can use here: (a) **trial and error** and (b) **use group and hope**. To get at either strategy, we need a good handle on the ipop as in

$$(2x+3)(3x-1) = 6x^2 + (9-2)x - 3 = 6x^2 + 7x - 3\,.$$

The Left and Right products take care of themselves, the sign analysis as above still applies, but the ipop is our main concern. All the coefficients and constants below are positive:

(D) $(Ax+a)(Bx+b) = ABx^2 + (aB + Ab)x + ab$

(E) $(Ax-a)(Bx-b) = ABx^2 - (aB + Ab)x + ab$

(F) $(Ax+a)(Bx-b) = ABx^2 + (aB - Ab)x - ab$

(6a) In the **trial and error method**, we search for numbers ***A***, ***B***, ***a*** and ***b*** where ***AB*** = quadratic coefficient, ***ab*** = constant and the ipop works out to be the linear term.
Some examples.

$$6x^2 - 7x + 2 = (Ax-a)(Bx-b) \qquad AB = 6,\ ab = 2,\ aB + Ab = 7$$

trying the various factors of **6** and **2** leads to $1\cdot 3 + 2\cdot 2 = 7$

$$= (2x-1)(3x-2)$$

$$3x^2 + 14x + 8 = (Ax+a)(Bx+b) \qquad AB = 3,\ ab = 8,\ aB + Ab = 14$$

trying the possibilities leads to $2\cdot 1 + 4\cdot 3 = 14$

$$= (3x+2)(x+4)$$

$$6x^2 - 7x - 3 = (Ax+a)(Bx-b) \qquad AB = 6,\ ab = 3,\ aB - Ab = -7$$

keep trying: $1\cdot 2 - 3\cdot 3 = -7$

$$= (3x+1)(2x-3)$$

(6b) In the **use group and hope method** we need to take a closer look at the (*D*), (*E*) and (*F*) forms above. The linear coefficient in each case is the sum or difference of ***aB*** and ***Ab***. Note that their product [(***aB***)(***Ab***)=(***AB***)(***ab***)] is the product of the leading coefficient and the constant. Thus, if we form the product of the leading coefficient and the constant, **(*AB*)(*ab*)**, we try to find two numbers having this product and whose (a) <u>sum</u> is the linear coefficient in forms (*D*) and (*E*) or whose (b) <u>difference</u> is the linear coefficient in form (*F*).
Some examples.

$6x^2 - 7x + 2$	$6 \cdot 2 = 12$ so we need to find two numbers whose product is **12** and whose sum is **7**. **3** and **4** do it.
$6x^2 - 3x - 4x + 2$	Group and hope
$3x(2x-1) - 2(2x-1)$	Factor out $2x-1$
$(3x-2)(2x-1)$	

$3x^2 + 14x + 8$	$3 \cdot 8 = 24$ so we need to find two numbers whose product is **24** and whose sum is **14**. **2** and **12** do it.
$3x^2 + 2x + 12x + 8$	Group and hope
$x(3x+2) + 4(3x+2)$	Factor out $3x+2$
$(x+4)(3x+2)$	

$6x^2 - 7x - 3$	$6 \cdot 3 = 18$ so we need to find two numbers whose product is **18** and whose DIFFERENCE is -7. $2 - 9 = -7$ works.
$6x^2 + 2x - 9x - 3$	Group and hope
$2x(3x+1) - 3(3x+1)$	Factor out $3x+1$
$(2x-3)(3x+1)$	