

Radicals

The Basics

(0) In this discussion we will ONLY deal with REAL quantities – that is, no Complex Numbers

(1) The **index** for any radical can only be a positive integer greater than 1. For such indices, we make these definitions:

If n is odd, $\sqrt[n]{x} = b \text{ where } b^n = x.$	If n is even and $x \geq 0$, $\sqrt[n]{x} = b \text{ where } b \geq 0 \text{ and } b^n = x.$
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For odd indices x can be positive, zero or negative and the resulting $\sqrt[n]{x}$ will be positive, zero or negative respectively. For example, $\sqrt[3]{27} = 3$, $\sqrt[7]{0} = 0$, $\sqrt[5]{-32} = -2$. However, for even indices the radicand MUST be non-negative and the resulting $\sqrt[n]{x}$ will also be non-negative. For example, $\sqrt[4]{16} = 2$, but $\sqrt[6]{-64}$ is not defined. (Actually, it can be defined, but it involves i , and we've agreed not to use i here.)

(4) We have another notation for $\sqrt[n]{x}$ that uses exponents: $\sqrt[n]{x} = x^{\frac{1}{n}}$. These fractional exponents obey all the laws of exponents. For example, “when you multiply the bases, you add the exponents”:

$x^{\frac{1}{2}} \cdot x^{\frac{1}{3}} = x^{\frac{1}{2} + \frac{1}{3}} = x^{\frac{5}{6}}$. However, we haven't defined objects like $x^{\frac{5}{6}}$ yet so we won't push this idea here. So far we've only defined the meaning of fractional exponents if the numerator is 1 (a **unit fraction**).

(5) On your calculator you use parentheses around the exponent. For example, $\sqrt[3]{8} = 8^{\frac{1}{3}}$ is entered on the calculator as $8^{(1 \div 3)}$. Similarly, $\sqrt[5]{-243} = (-243)^{\frac{1}{5}}$ would be entered as $(-243)^{(1 \div 5)}$.

(6) We have the following rule.

If n is odd, $\sqrt[n]{x^{tn}} = x^t.$	If n is even and $x \geq 0$, $\sqrt[n]{x^{tn}} = x^t.$
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These rules make sense if you look at the exponent form: $\sqrt[n]{x^{tn}} = (x^{tn})^{\frac{1}{n}} = x^{\frac{tn}{n}} = x^t$. Thus, you can eliminate the radical if the radicand is an object raised to a power which is a multiple of the index. For example,

$$\sqrt[3]{x^{12}} = x^{\frac{12}{3}} = x^4 \text{ and } \sqrt[4]{16x^8y^{20}} = (2^4x^8y^{20})^{\frac{1}{4}} = 2x^2y^5.$$

(7) Note: $\sqrt{x^2} = |x|$. This is obvious if x is non-negative: $\sqrt{3^2} = \sqrt{9} = 3 = |3|$. It really comes into play if x is negative: $\sqrt{(-5)^2} = \sqrt{25} = 5$ which is NOT what we started with. In this case $5 = |-5|$ which is what the rule requires.

Some Manipulative Skills

(1) A basic definition:

<p>If m and n are integers with $n > 1$, then</p> $x^{\frac{m}{n}} = \left\{ \begin{array}{l} (x^n)^{\frac{1}{n}} = (\sqrt[n]{x})^m \\ (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m} \end{array} \right\}$ <p>provided $\sqrt[n]{x}$ is real.</p>
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We have the option on $x^{\frac{m}{n}}$ to (a) root first, then power or (b) power first, then root. However, the “root first, then power” part of the definition is the **ONLY WAY** we want to do these things. It is in keeping with the idea of

make small before making big that we saw long ago. So, for example: $8^{\frac{4}{3}} = (\sqrt[3]{8})^4 = 2^4 = 16$ or

$$9^{-\frac{3}{2}} = \frac{1}{9^{\frac{3}{2}}} = \frac{1}{(\sqrt{9})^3} = \frac{1}{3^3} = \frac{1}{27}$$

(2) Other manipulations:

(a) $\frac{(x^{\frac{1}{2}} y^{\frac{2}{3}})^4}{y^2}$ mult. & div. only \Rightarrow put power through the product

$$\frac{x^2 y^{\frac{8}{3}}}{y^2}$$

divide the bases \Rightarrow subtract the exponents

$$x^2 y^{\frac{8}{3} - 2} = x^2 y^{\frac{16-3}{6}}$$

$$x^2 y^{\frac{13}{6}}$$

(b) $\frac{(x^{-\frac{1}{6}})^3}{x^{\frac{1}{3}} x^{-\frac{2}{3}} x^{\frac{1}{6}}}$ mult. & div. only \Rightarrow flip LAST

$$\frac{x^{-\frac{1}{2}}}{x^{\frac{1}{3} - \frac{2}{3} + \frac{1}{6}}} = \frac{x^{-\frac{1}{2}}}{x^{-\frac{1}{6}}}$$

divide the basis \Rightarrow subtract the exponents

$$x^{-\frac{1}{2} - (-\frac{1}{6})} = x^{-\frac{1}{2} + \frac{1}{6}} = x^{-\frac{1}{3}}$$

now flip

$$\frac{1}{x^{\frac{1}{3}}}$$

$$(c) \sqrt[3]{x^2} \sqrt[4]{x}$$

$$x^{\frac{2}{3}} x^{\frac{1}{4}} = x^{\frac{2}{3} + \frac{1}{4}}$$

$$x^{\frac{11}{12}} = \sqrt[12]{x^{11}}$$

indices different \Rightarrow convert to fractional exponents

(3) More rules:

$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$	(M1)
$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	(M2)
provided at least one of $\sqrt[n]{a}$ and $\sqrt[n]{b}$ is real	

Remember:

There is NO RULE for simplifying a radicand that is a sum or difference:

$$\sqrt[n]{A + B} \neq \sqrt[n]{A} + \sqrt[n]{B}$$

We combine the rules above with the following rule to not only manipulate but also simplify radicals:

$\sqrt[n]{a^{tn}} = a^t$	(S)
provided $a \geq 0$ if n is even	

For example, to simplify $\sqrt[3]{16x^{13}y^{11}} = \sqrt[3]{2^4 x^{13} y^{11}}$ we begin by finding the largest multiples of **3** that do not exceed the given exponents: **3**, **12**, and **9** respectively in this case. We “peel these powers off”

$\sqrt[3]{(2^3 \cdot x^{12} y^9)(2xy^2)}$. Now, we can apply (S) along with (M1) all at once after we note that $\frac{3}{3} = 1$, $\frac{12}{3} = 4$ and

$$\frac{9}{3} = 3: 2^1 x^4 y^3 \sqrt[3]{2xy^2} = 2x^4 y^3 \sqrt[3]{2xy^2}.$$

Rationalizing the Denominator

(1) The process of eliminating radicals from denominators is called **rationalizing the denominator**. There are many techniques for doing this depending on the nature of the denominator. We’ll study only two of them. The first of these methods applies when the denominator is just a radical, and it uses the insight gained by having worked with the (S) rule above.

Examples:

(a) $\frac{3}{\sqrt{5}}$

Since this is a square root, we need objects in the radicand that are raised to a

power that is a multiple of **2**. We have 5^1 , so 5^2 is the closest. We can get that by multiplying numerator and denominator by $\sqrt{5}$. I like to think of this as “multiply by **1** in disguise with radicals”—the old Beatles song!

$$\frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{\sqrt{25}} = \frac{3\sqrt{5}}{5}$$

(b) $\sqrt{\frac{3m^3}{p^5t^4}}$ First use (M2).

$$\frac{\sqrt{3m^3}}{\sqrt{p^5t^4}}$$

4 is a multiple of 2, so t^4 is not a worry. The next multiple of 2

past 5 is 6, so we multiply by 1 in disguise with radicals: $\frac{\sqrt{p}}{\sqrt{p}}$. While

we're at it, we'll simplify the numerator.

$$\frac{m\sqrt{3m}}{\sqrt{p^5t^4}} \cdot \frac{\sqrt{p}}{\sqrt{p}} = \frac{m\sqrt{3mp}}{\sqrt{p^6t^4}} = \frac{m\sqrt{3mp}}{p^3t^2}$$

(c) $\frac{1}{\sqrt{xy^2z^7}}$ This is getting boring. The current powers are 1, 2, and 7. So

we need 2, 2, and 8 which we can get by using \sqrt{xz} .

$$\frac{1}{\sqrt{xy^2z^7}} \cdot \frac{\sqrt{xz}}{\sqrt{xz}} = \frac{\sqrt{xz}}{\sqrt{x^2y^2z^8}} = \frac{\sqrt{xz}}{xyz^4}$$

(d) $\frac{1}{\sqrt[3]{xy^2z^7}}$ Oh, the same radicand, but a new index. Now we want powers

that are multiples of 3. In this case we want 3, 3, and 9. Thus we use

$$\frac{1}{\sqrt[3]{xy^2z^7}} \cdot \frac{\sqrt[3]{x^2yz^2}}{\sqrt[3]{x^2yz^2}} = \frac{\sqrt[3]{x^2yz^2}}{\sqrt[3]{x^3y^3z^9}} = \frac{\sqrt[3]{x^2yz^2}}{xyz^3}$$

(e) $\frac{1}{\sqrt[4]{xy^2z^7}}$ Now the index is 4, so we need the powers 4, 4, and 8.

$$\frac{1}{\sqrt[4]{xy^2z^7}} \cdot \frac{\sqrt[4]{x^3y^2z}}{\sqrt[4]{x^3y^2z}} = \frac{\sqrt[4]{x^3y^2z}}{\sqrt[4]{x^4y^4z^8}} = \frac{\sqrt[4]{x^3y^2z}}{xyz^2}$$

As we see from these last three examples, what we multiply the numerator and denominator by depends both on the radicand and the index.

(2) Before we can introduce the second technique, we need to do some radical arithmetic. The pairs of objects like $\{3 + \sqrt{2}, 3 - \sqrt{2}\}$ and $\{\sqrt{5} - \sqrt{t}, \sqrt{5} + \sqrt{t}\}$ are called **conjugate** pairs. If we apply the special product $(A + B)(A - B) = A^2 - B^2$ to these, we get:

$$(3 + \sqrt{2})(3 - \sqrt{2}) = 3^2 - (\sqrt{2})^2 = 9 - 2 = 7$$

$$(\sqrt{5} - \sqrt{t})(\sqrt{5} + \sqrt{t}) = (\sqrt{5})^2 - (\sqrt{t})^2 = 5 - t$$

Notice that the radicals vanish in this process. This observation is used to handle fractions whose denominators are one of these types of objects. Some examples follow. NOTE: The above ONLY applies for conjugates involving SQUARE ROOTS – it is not true for cube roots, etc.

(a) $\frac{2}{\sqrt{3}-1}$ Use $\sqrt{3}+1$ and multiply by 1 in disguise with radicals.

$$\frac{2}{\sqrt{3}-1} \cdot \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{2(\sqrt{3}+1)}{3-1} = \frac{2(\sqrt{3}+1)}{2} = \sqrt{3}+1$$

(b) $\frac{5}{4-\sqrt{2}}$ Use $4+\sqrt{2}$.

$$\frac{5}{4-\sqrt{2}} \cdot \frac{4+\sqrt{2}}{4+\sqrt{2}} = \frac{5(4+\sqrt{2})}{16-2} = \frac{5(4+\sqrt{2})}{14}$$

(c) $\frac{3}{\sqrt{5}+\sqrt{3}}$ Use $\sqrt{5}-\sqrt{3}$.

$$\frac{3}{\sqrt{5}+\sqrt{3}} \cdot \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \frac{3(\sqrt{5}-\sqrt{3})}{5-3} = \frac{3(\sqrt{5}-\sqrt{3})}{2}$$

Radical Equations

(1) A **radical equation** is any equation containing a radical with the variable we want to solve for as part of a radicand. At this level, we will only be interested in square roots – although once you understand the method for square roots, you should be able to extend its basic idea to other indices, but I won't ask you to do that here.

(2) Solving radical equations (square root ones) is fairly straight forward and the process sounds like the bar tender on the old Love Boat: **ISSC**.

Isolate a radical
 Square both sides of the equation
 Solve the resulting equation
 Check your solutions

Example 1:

$$\sqrt{4-x} - 2 = x$$

Isolate a radical.

$$\sqrt{4-x} = x + 2$$

Square both sides.

$$(\sqrt{4-x})^2 = (x+2)^2$$

Note: Square both SIDES, not square all terms!!!!

$$4-x = x^2 + 4x + 4$$

Solve the resulting equation.

Quadratic \Rightarrow Make equation to 0

$$x^2 + 5x = 0$$

Factor.

$$x(x+5) = 0$$

Zero Product Rule.

$$x = 0 \text{ or } x = -5$$

But we're not done just yet! The C in ISSC says Check your solutions. Check them, that is, IN THE ORIGINAL EQUATION $\sqrt{4-x} - 2 = x$. Watch *very carefully* what happens in the following checking process. We substitute each "answer" into the ORIGINAL EQUATION and ask if we do indeed have equality.

$x = 0$

$\sqrt{4-0} - 2 \text{ ?? } 0$

$\sqrt{4} - 2 \text{ ?? } 0$

$2 - 2 \text{ ?? } 0$

$0 = 0 \text{ OK}$

$x = -5$

$\sqrt{4-(-5)} - 2 \text{ ?? } -5$

$\sqrt{4+5} - 2 \text{ ?? } -5$

$\sqrt{9} - 2 \text{ ?? } -5$

$3 - 2 \text{ ?? } -5$

$1 \neq -5 \text{ Bummer!!!!}$

So $x = -5$ does NOT check, and hence it must be tossed out. The only solution is $x = 0$.

What happened here is that $x = 0$ and $x = -5$ are, indeed, solutions to $4 - x = x^2 - 4x + 4$, the “squared” equation, but only $x = 0$ is a solution to the original equation. The quantity -5 is called an **extraneous root: a solution to the square of an equation which is NOT a solution to the original equation**. This can often happen and hence, along with fractional equations, the technique for solving radical equations requires checking. Unlike the simple checking for zero denominators that was required for fractional equations, the checking here must be complete – substitute into the ORIGINAL EQUATION and verify whether or not the two sides reduce to the same value. In general, anything could happen in this checking process: you might have to throw one or more solutions away (as above), throw them all away (solution set is then \emptyset), or you may get to keep them all.

Example 2:

$\sqrt{5x+6} + \sqrt{3x+4} = 2$

Isolate a radical, say the first one.

$\sqrt{5x+6} = 2 - \sqrt{3x+4}$

Square both sides.

$(\sqrt{5x+6})^2 = (2 - \sqrt{3x+4})^2$

$5x + 6 = 4 - 4\sqrt{3x+4} + 3x + 4$

Solve the resulting equation. But this is a radical equation \Rightarrow ISSC. So Isolate a radical.

$5x + 6 = 8 + 3x - 4\sqrt{3x+4}$

$4\sqrt{3x+4} = 2 - 2x$

Divide through by 2 to simplify.

$2\sqrt{3x+4} = 1 - x$

Square both sides.

$(2\sqrt{3x+4})^2 = (1 - x)^2$

$4(3x+4) = 1 - 2x + x^2$

Note the $2^2 = 4$ up front.

Solve the resulting equation.

$12x + 16 = 1 - 2x + x^2$

Quadratic \Rightarrow make equation to 0.

$x^2 - 14x - 15 = 0$

Factor.

$(x - 15)(x + 1) = 0$

Zero Product Rule.

$x = 15 \text{ or } x = -1$

Check.

$x = 15$

$\sqrt{5(15)+6} + \sqrt{3(15)+4} \text{ ?? } 2$

 \vdots

$16 \neq 2 \text{ Throw out}$

$x = -1$

$\sqrt{5(-1)+6} + \sqrt{3(-1)+4} \text{ ?? } 2$

 \vdots

$2 = 2 \text{ OK}$

Thus the solution is $x = -1$.