Rational Expressions

Simplifying

(1) The **Cancellation Law** allows us to cancel FACTORS (not TERMS):

$$\text{If } c \neq 0, \text{ then } \frac{a \bullet c}{b \bullet c} = \frac{a}{b}.$$

REMEMBER THAT THE OBJECT YOU WANT TO CANCEL **MUST** BE A **FACTOR** OF THE ENTIRE NUMERATOR AND THE ENTIRE DENOMINATOR. YOU CANNOT CANCEL TERMS.

The x^2 in $\frac{x^2 - 3x}{x^2 - x + 6}$ does **not** cancel because it is not a factor of either the numerator nor the denominator. However, the x^2 in $\frac{5x^2}{x^2(x-4)}$ does cancel since it is a factor of both the numerator and the denominator. The Examples below show this process at work.

(2) Examples:

(a) $$\frac{x^4 - x^2}{x^3 + 2x^2 + x} = \frac{x^2(x^2-1)}{x(x^2+2x+1)} = \frac{x^2(x+1)(x-1)}{x(x+1)^2} = \frac{x \cdot \cancel{x}\,\cancel{(x+1)}(x-1)}{\cancel{x}\,\cancel{(x+1)}(x+1)} = \frac{x(x-1)}{x+1}$$

Notice the GCF factoring first, then a difference of squares on top, and a perfect square trinomial on the bottom. (The fourth fraction above typically isn't written in an actual problem unless you are having a lot of difficulty canceling properly – I wrote it out for clarity purposes here only.)

(b) $$\frac{16 - 4x^2}{x^3 - x^2 - 2x} = \frac{4(4-x^2)}{x(x^2-x-2)} = \frac{4(2+x)(2-x)}{x(x-2)(x+1)} = \frac{4(2+x)}{x(x+1)} \cdot \frac{2-x}{x-2} = \frac{4(2+x)}{x(x+1)}(-1) = \frac{-4(x+2)}{x(x+1)}$$

Here we have $\frac{2-x}{x-2}$ simplifying to **–1**. Notice how we left the denominator in factored form. There is never a good reason to multiply out a denominator that has already been factored. For now you can either multiply out the numerator or not (as you wish), but NEVER MULTIPLY OUT DENOMINATORS.

Multiply and Divide

(1) Multiplication and division of fractions follow the basic arithmetic rules you know:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

The division rule is often referred to as "invert and multiply."

(2) **Make Small Before You Make Big**. I wish this suggestion would be given more emphasis in grade school. For example, too many people multiply $\frac{12}{35}$ by $\frac{21}{8}$ like this: $\frac{12}{35} \cdot \frac{21}{8} = \frac{252}{280}$ and then struggle to simplify the answer. Why make big first and then try to make small? **Make Small Before You Make Big** – that is, use the Cancellation Law EARLY like this: $\frac{12}{35} \cdot \frac{21}{8} = \frac{\cancel{4} \cdot 3}{5 \cdot \cancel{7}} \cdot \frac{\cancel{7} \cdot 3}{\cancel{4} \cdot 2} = \frac{9}{10}$. Always do whatever canceling you can, as early as you can. This is the key to making the transition to rational expressions from rational numbers.

(3) To multiply or divide algebraic fractions we simply use the general fraction rules augmented by the command: **Make Small Before You Make Big**. In order to do this, you must factor everything you can right from the start. For example:

$$\frac{x^2 - 5x + 6}{4x^3 - x} \cdot \frac{2x^2 - 5x - 3}{x^2 - 6x + 9} = \frac{(x-2)(x-3)}{x(4x^2-1)} \cdot \frac{(2x+1)\cancel{(x-3)}}{(x-3)^{\cancel{2}}} = \frac{(x-2)}{x\,\cancel{(2x+1)}(2x-1)} \cdot \frac{\cancel{(2x+1)}}{1} = \frac{x-2}{x(2x-1)}$$

Remember, don't multiply out factored denominators.

Another example:

$$\frac{x^2+3x}{x^2+x} \div \frac{9-x^2}{x^2-x-6} = \frac{x\,(x+3)}{x\,(x+1)} \cdot \frac{(x-3)(x+2)}{(3+x)(3-x)} = \frac{x+2}{x+1} \cdot \frac{x-3}{3-x} = \frac{x+2}{x+1} \cdot (-1) = \frac{-(x+2)}{x+1}$$

Notice that I saved some time by factoring in the divisor fraction in the invert and multiply step. Also notice the $\frac{x-3}{3-x} = -1$ simplification we saw before.

Add and Subtract

(1) There is only one rule for adding or subtracting fractions:

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \qquad \frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$$

This rule requires that both fractions have the same (or common) denominator as in: $\frac{3}{4} + \frac{7}{4} = \frac{10}{4} = \frac{5}{2}$. How then can we add fractions which do not have a common denominator? We use the **Fundamental Law of Fractions**

> If $x \neq 0$, then $\frac{a}{b} = \frac{ax}{bx}$.

to create a common denominator. For example, to add **3/4** to **1/6** we could select **48** as a common denominator:

$$\frac{3}{4} + \frac{1}{6} = \frac{3 \cdot 12}{4 \cdot 12} + \frac{1 \cdot 8}{6 \cdot 8} = \frac{36}{48} + \frac{8}{48} = \frac{44}{48} = \frac{11}{12}$$

Why did I pick **48**? Because it is legal – any common multiple of **4** and **6** would work nicely as a common denominator. However, if you don't use the *Least Common Multiple* of **4** and **6** as the common multiple, then the resulting fraction will NEVER be in simplified form – you will ALWAYS have to worry about simplifying it. That is why we like to use the LCM of the denominators as the common denominator – the resulting fraction usually (but not always) will be in simplified form immediately.

(2) Before we look at the process, a word about terminology. The Least Common Denominator of $\frac{a}{b}$ and $\frac{c}{d}$ is the Least Common Multiple of b and d. The LCD refers to the fractions themselves, while the LCM refers to the denominators. So, $\text{LCD}\left(\frac{a}{b}, \frac{c}{d}\right) = \text{LCM}(b, d)$. In the above example, $\text{LCD}\left(\frac{3}{4}, \frac{1}{6}\right) = \text{LCM}(4, 6) = 12$.

(3) In 4th and 5th grades you learned a few ways to find the LCM of two whole numbers. The following is one of them and the only one that is efficient when we make the transition to algebraic fractions. I'll demonstrate with **20** and **24**.

(a) *Factor completely – using exponents when possible.*	$20 = 2^2 \cdot 5$ and $24 = 2^3 \cdot 3$
(b) *Write down once every distinct prime factor you see.*	$2 \cdot 3 \cdot 5$
(c) *Raise each factor to the highest power seen on it.*	$2^3 \cdot 3^1 \cdot 5^1 = 2^3 \cdot 3 \cdot 5 = 120$

(4) This is the three step process for finding the LCM of any finite number of expressions. *Know this process cold*. Let's do an algebraic example and find the LCM of $x-1$, x^2-1 and x^2+2x+1.

(a) *Factor completely – using exponents when possible.*

$$x-1 = x-1 \qquad x^2-1 = (x+1)(x-1) \qquad x^2+2x+1 = (x+1)^2$$

(b) *Write down once every distinct prime factor you see.*

$$(x-1)(x+1)$$

(c) *Raise each factor to the highest power seen on it.*

$$(x-1)(x+1)^2$$

(5) OK, so now that we can find the LCD of a number of fractions, how do we use it to add them?

Using the LCD as the denominator of the resulting fraction, its numerator is obtained by taking each fraction in turn and multiplying its numerator by the factors of the LCD missing from its denominator.

Say what? Let's illustrate with $\frac{3}{20}+\frac{7}{24}$. Recall $20=2^2\cdot 5$ and $24=2^3\cdot 3$ and the LCD is $2^3\cdot 3\cdot 5$. To make the LCD the denominator of $\frac{3}{20}=\frac{3}{2^2\cdot 5}$ we need to multiply numerator and denominator by $2\bullet 3$: $\frac{3\cdot(2\cdot 3)}{2^2\cdot 5\cdot(2\cdot 3)}=\frac{18}{2^3\cdot 3\cdot 5}$. To make the LCD the denominator of $\frac{7}{24}=\frac{7}{2^3\cdot 3}$ we need to multiply numerator and denominator by 5:

$\frac{7\cdot(5)}{2^3\cdot 3\cdot(5)}=\frac{35}{2^3\cdot 3\cdot 5}$. So we have:

$$\frac{3}{20}+\frac{7}{24}=\frac{18}{2^3\cdot 3\cdot 5}+\frac{35}{2^3\cdot 3\cdot 5}=\frac{18+35}{2^3\cdot 3\cdot 5}=\frac{53}{2^3\cdot 3\cdot 5}$$

Let's look at the second to last fraction above in a different way: $\frac{18+35}{2^3\cdot 3\cdot 5}=\frac{3\cdot(2\cdot 3)+7\cdot(5)}{2^3\cdot 3\cdot 5}$. This is a fraction whose denominator is the LCD and whose numerator comes from multiplying the numerator of each fraction by the factors of the LCD missing from its denominator. Hence the technique in the box above.

(6) Let's find $\frac{2}{x-1}+\frac{3}{x^2-1}-\frac{1}{x^2+2x+1}$. Begin by factoring the denominators.

$$\frac{2}{x-1}+\frac{3}{x^2-1}-\frac{1}{x^2+2x+1}=\frac{2}{x-1}+\frac{3}{(x+1)(x-1)}-\frac{1}{(x+1)^2}$$

The LCD is $(x-1)(x+1)^2$ by the three step process. This will be the denominator of the sum:

$$\frac{*}{(x-1)(x+1)^2}$$

To get the numerator, we compare the denominator of each fraction with the LCD:

$\frac{2}{x-1}\Leftarrow$ missing from the denominator is $(x+1)^2$, so we start our numerator with the product of 2 and this missing piece: $\frac{2(x+1)^2+*}{(x-1)(x+1)^2}$

$\frac{3}{(x+1)(x-1)}\Leftarrow$ missing from the denominator is $x+1$, so we continue our numerator with the product of 3 and this missing piece: $\frac{2(x+1)^2+3(x+1)-*}{(x-1)(x+1)^2}$

$\frac{1}{(x+1)^2}\Leftarrow$ missing here is $x-1$, so we finish our numerator with the product of 1 (or technically -1) and this missing piece: $\frac{2(x+1)^2+3(x+1)-1(x-1)}{(x-1)(x+1)^2}$

Thus, we get $\dfrac{2(x+1)^2+3(x+1)-1(x-1)}{(x-1)(x+1)^2}$ which we proceed to simplify (i.e., expand the numerator):

$$\frac{2(x^2+2x+1)+3x+3-x+1}{(x-1)(x+1)^2}=\frac{2x^2+4x+2+2x+4}{(x-1)(x+1)^2}=\frac{2x^2+6x+6}{(x-1)(x+1)^2}=\frac{2(x^2+3x+3)}{(x-1)(x+1)^2}$$

We had to expand the numerator to see if it could be factored just in case the big fraction was not in simplified form.

(7) Whenever the denominators have no common factor, the LCD is just the product of the denominators. The LCM of two expressions having no common factors is just their product. For example:
$\dfrac{3}{4}+\dfrac{7}{15}=\dfrac{3\cdot 15+4\cdot 7}{4\cdot 15}=\dfrac{45+28}{60}=\dfrac{73}{60}$. This also applies to any fraction plus a whole thing as we've seen before:
$\dfrac{3}{4}+5=\dfrac{3}{4}+\dfrac{5}{1}=\dfrac{3\cdot 1+4\cdot 5}{4\cdot 1}=\dfrac{3+20}{4}=\dfrac{23}{4}$. You must be very quick and very good with problems like these:

(a) $\dfrac{1}{x-2}-\dfrac{4}{x+3}=\dfrac{x+3-4(x-2)}{(x-2)(x+3)}=\dfrac{x+3-4x+8}{(x-2)(x+3)}=\dfrac{11-3x}{(x-2)(x+3)}$

(b) $\dfrac{x}{x+1}+\dfrac{3}{x}=\dfrac{x^2+3(x+1)}{x(x+1)}=\dfrac{x^2+3x+3}{x(x+1)}$

(c) $\dfrac{3}{x+5}-2=\dfrac{3}{x+5}-\dfrac{2}{1}=\dfrac{3-2(x+5)}{x+5}=\dfrac{3-2x-10}{x+5}=\dfrac{-2x-7}{x+5}$

(d) $4-\dfrac{x}{x+1}=\dfrac{4}{1}-\dfrac{x}{x+1}=\dfrac{4(x+1)-x}{x+1}=\dfrac{4x+4-x}{x+1}=\dfrac{3x+4}{x+1}$

Note very carefully in examples (a) and (c) above how the -4 and -2 respectively get distributed in the numerator. In example (a) we have: $x+3-4(x-2)=x+3-4x-4(-2)=x+3-4x+8$; many students forget to carry the negative along when multiplying -4 by -2.