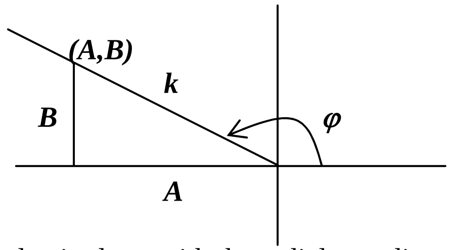
Sums of the form Asin(x) + Bcos(x)

Consider any pair of numbers ***(A,B)*** with neither ***A*** nor ***B*** being **0.** We'll use a 2nd quadrant pair to illustrate our development.



$$k = \sqrt{A^2 + B^2}$$

This ordered pair along with the radial arm distance, ***k***, determine an angle φ having the property that

$$\sin\varphi = \frac{B}{k} \quad \text{and} \quad \cos\varphi = \frac{A}{k}$$

With that in mind, let's play with the quantity $A\sin x + B\cos x$ for any value x:

$$\begin{aligned} A\sin x + B\cos x &= k\cdot\frac{A}{k}\sin x + k\cdot\frac{B}{k}\cos x = k\left(\frac{A}{k}\sin x + \frac{B}{k}\cos x\right) \\ &= k\left(\cos\varphi\sin x + \sin\varphi\cos x\right) = k\left(\sin x\cos\varphi + \cos x\sin\varphi\right) \\ &= k\sin\left(x+\varphi\right) \end{aligned}$$

Thus we are able to write $A\sin x + B\cos x$ as a function involving the sine function of a new value $x+\varphi$. This is a skill that is needed at certain stages in the calculus and physics.

> If neither ***A*** nor ***B*** is zero, then $A\sin x + B\cos x$ can be written as $k\sin(x+\varphi)$ where $k = \sqrt{A^2+B^2}$ and φ is the <u>unique</u> angle satisfying the three conditions:
>
> (i) $-\pi < \varphi \le \pi$ (ii) $\sin\varphi = \frac{B}{k}$ (iii) $\cos\varphi = \frac{A}{k}$

Example 1

$$\sqrt{3}\sin x - \cos x$$

Solution:

$A=\sqrt{3}$ and $B=-1$, so $k=\sqrt{3+1}=\sqrt{4}=2$ and the ordered pair $(\sqrt{3},-1)$ is in the 4th quadrant. Thus, $\sin\varphi = \frac{-1}{2}$ and $\cos\varphi = \frac{\sqrt{3}}{2}$, and the reference angle is $\varphi_R = \frac{\pi}{6}$. So, $\varphi = \frac{-\pi}{6}$ to keep in the $-\pi < \varphi \le \pi$ range. We then have $\sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right)$.

Example 2

$$-2\sin x + 5\cos x$$

Solution:

$A=-2$ and $B=5$, so $k=\sqrt{4+25}=\sqrt{29}$. The ordered pair $(-2,5)$ is in the 2nd quadrant. Thus, $\sin\varphi = \frac{5}{\sqrt{29}}$ and $\cos\varphi = \frac{-2}{\sqrt{29}}$. Using the calculator gives roughly $\varphi_R = \sin^{-1}\left(\frac{5}{\sqrt{29}}\right) \approx 1.19$. So, $\varphi = \pi - 1.19 \approx 1.95$. We then have $-2\sin x + 5\cos x = \sqrt{29}\sin(x+1.95)$.